An Attack on Flexibility and Stoker's Problem

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Abstract

In view of solving questions of geometric realizability of polyhedra under given geometric constraints, we parametrize the moduli-space of simply connected polyhedra in terms of their face and their dihedral angles. We recompute its dimension at points of smoothness modulo a combinatorial conjecture.

Contents

1 Introduction

Polyhedra are simple objects that everybody knows and that are widely used. Nevertheless, fundamental questions about their shape remain open. For instance, we would like to know how the shape of a polyhedron depends on the shape of its faces, and this seemingly simple question still eludes us. It is easy to come up with examples of polyhedra, that show that the shape of the faces does not determine the shape of the polyhedron, see Figure 1. Looking

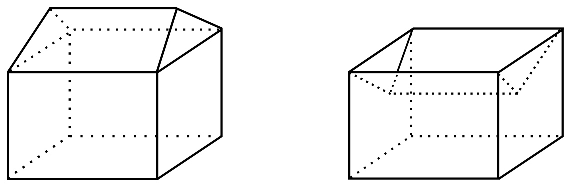


Figure 1: Two houses, one with a roof, and one with a roof pool.

at the examples in Figure 1, and similar ones, we might be led to think that the only non-uniqueness of the shape of a polyhedron comes about by reflections through planes containing a closed edge-path. In particular, there would be only finitely many shapes of a polyedron with given combinatorics and faces. However, one can even construct continuous families of such, one of which is shown in Figure 2.

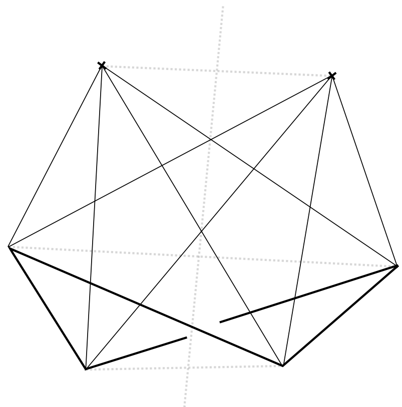


Figure 2: Bricard's octahedron of type I intersects itself.

We call such polyhedra *flexible*, and the above example along with two other types was the first one found [3]. With some more maneuvering one can also produce examples of flexible polyhedra without self-intersections, which was done by Connelly [5, 6] in the late seventies. In Figure 3, Steffen's

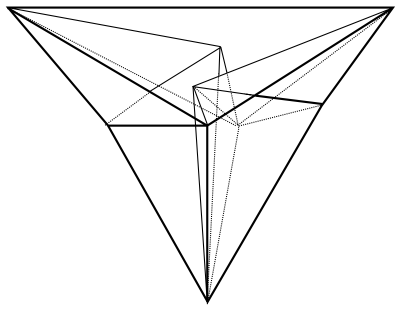


Figure 3: Steffen's polyhedron.

polyhedron consists of two Bricard octahdera with two removed faces and joined by a flap. A printable folding pattern can be found on cutupfoldup.com.

The question whether a given polyhedron is flexible is asking, whether there is a continuous family of realizations for given fixed face angles. Similarly, one might ask whether the dihedral angles of a polyhedron determine its shape, and again we know very little. We do know that there are polyhedra with the same dihedrals and different shapes [?], and that infinitesimally the shape cannot change when the dihedrals are fixed [12]. We also know that the shape is unique for certain restrictive subclasses of convex polyhedra with fixed dihedrals [10, 13, 14]. The question whether this is true for all convex polyhedra is known as Stoker's Conjecture.

Indeed, one is often led to ask for any set of given geometric data, such as subsets of face angles, dihedrals or edgelengths, what the space of possible realizations is. In the following we attack these questions by giving a new description of the space of shapes that is natural to the problem. We note that spaces of shapes of polyhedra have been studied, see [15], and references therein, and Whiteley [17]. Here, the space of shapes is described by studying the possible liftings to three-space of planar graphs that may be considered projections of polyhedra, and several conlcusions about which geometric data can be determined from which other data have been drawn from it. In [16] Thurston Sr. studies the space of flat-cone surfaces with given convex cone-singularities, each of which by a theorem of Alexandrov [1] and by Cauchy's Rigidity Theorem [4, 8, 11] can be uniquely realized as a convex polyhedron. However, this proof is non-constructive, so that it does not help to relate geometric contraints. In the following we describe the shape of a polyhedron in terms of its face angles and its dihedral angles by describing it as a flat-cone surface with specified triangulation and by describing the polyhedral cones defined around each vertex.

By a polyhedron we really mean the following.

Definition 1. A *combinatoric* is a nonempty, consistently oriented abstract simplicial two-complex, whose geometric realization is homeomorphic to a closed connected surface.

Definition 2. A *polyhedron* is a simplexwise linear and simplexwise injective map $P\colon |K| \to \mathbb{R}^3$ from the geometric realization $|K|$ of a combinatoric K into euclidean $\mathbb{R}^3$.

We call the images of a polyhedron of its zero-, one- and two-simplices its *vertices*, *edges*, and *faces*, respectively. Their numbers will always be denoted by the letters V, E and F, respectively. Note that our polyhedra are always triangulated and oriented. Furthermore, a polyhedron may intersect itself and even degenerate into a plane, but its faces and edges are non-degenerate. We call two faces or edges *adjacent*, whenever their preimages are adjacent as simplices. A polyhedron $P\colon |K| \to \mathbb{R}^3$ defines two special sets of angles, namely

$$\delta_P \in [0, 2\pi)^{K_1} \qquad \text{and} \qquad \sigma_P \in (0, \pi)^{C_K},$$

where δ_K are the dihedral angles between adjacent faces of P, labeled by the one-simplices of K, and σ_K are the angles of the faces of P labeled by the set of *corners of K*

$$C_K := \big\{(ijk) \mid (ijk) \text{ is an order of } \{ijk\} \in K_2\big\} \Big/ \sim,$$

where $\sim$ identifies an order (ijk) with the order (kji).

It is easy to prove that the angles δ_P and σ_P completely determine the shape of a polyhedron, so that they are natural coordinates for a space of shapes. To define such a space we would like to know which vectors $\sigma \in (0,\pi)^{C_K}$ and $\delta \in [0, 2\pi)^{K_1}$ are the surface and dihedral angles of a polyhedron. To answer this question we observe that every polyhedron defines a triangulated flat-cone surface. In Section 2, we parametrize the space $\mathcal{T}_K^{\text{in}}$ of flat-cone surfaces that are triangulated by a combinatoric K in terms of its surface angles, by parametrizing the space of euclidean triangles for each of the faces. In addition, we note that every polyhedron defines a polyhedral cone around each of its vertices. In Section 3.2 we parametrize the space $\mathcal{T}_n$ of polyhedral cones with n edges in terms of its surface and its dihedral angles at smooth points. We note that Kapovich and Millson [9] have allready studied the space of polyhdral cones using Hodge Theory. We use a trick invented by Legendre, and intersect the polyhedral cone with a unit sphere centered at its vertex, see Figure 4. This intersection is a spherical polygon whose edgelengths are the surface angles of the polyhedral cone, and whose angles are the dihedral angles of the polyhedral cone. We study the

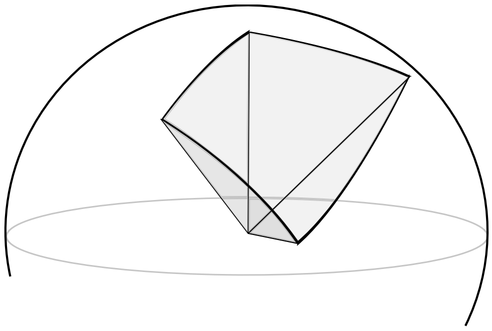


Figure 4: Legendre's trick

space of spherical n-gons by triangulating them and parametrizing the space of spherical triangles 3.1. We amass the spaces of polyhedral cones defined for each zero-simplex of a combinatoric K and identify dihedrals according to K into a space $\mathcal{T}_K^{\mathrm{le}}$ in Section 3.3. In summary, we observe that the surface and dihedral angles σ_P and δ_P of a polyhedron P are the surface angles of a flat-cone surface with triangulation K and the surface and dihedral angles of a certain collection of polyhedral cones that share dihedral angles at the one-simplices of K, i.e., that

$$(\sigma_P, \delta_P) \in \mathcal{T}_K := (\mathcal{T}_K^{\mathrm{in}} \times [0, 2\pi)^{K_1}) \cap \mathcal{T}_K^{\mathrm{le}}.$$

We find and prove in Section 4 that this is also sufficient for a set of angles to arise as those of a simply connected polyhedron P.

Theorem 3 (Main). *If K is a combinatoric such that $|K|$ is simply connected, then*

$$(\sigma, \delta) \in (0, \pi)^{C_K} \times [0, 2\pi)^{K_1}$$

are the surface and dihedral angles $\sigma = \sigma_P$ and $\delta = \delta_P$ of a polyhedron P if and only if $(\sigma, \delta) \in \mathcal{T}_K$.

Furthermore, we can compute some of the following dimensions.

Theorem 4. *At points of smoothness the spaces $\mathcal{T}_K^{in}$, $\mathcal{T}_n$, $\mathcal{T}_K^{le}$ and $\mathcal{T}_K$ are manifolds of the following dimensions*

$$\dim \mathcal{T}_n = 2n - 3, \quad \textit{and} \quad \dim \mathcal{T}_K^{le} = 2E + 6g - 6,$$

where g is the genus of $|K|$, and it is well known and we recompute modulo a well decorated conjecture 34, that

$$\dim \mathcal{T}_K = \dim \mathcal{T}_K^{in} = E - 1.$$

2 Parametrizing intrinsic constraints

Intrinsically, a polyhedron is a *flat-cone surface*, that is, with the induced metric from the ambient euclidean three-space, it is a surface that is locally isometric to the euclidean plane, with the exception of neighborhoods of finitely many points that are isometric to a euclidean cone. Furthermore, the edges of $P\colon |K| \to \mathbb{R}^3$ define a triangulation of its flat-cone surface S_P, whose vertices are the cone points of S_P. We note that P defines a map $\ell_P \in \mathbb{R}_+^{K_1}$ of edgelengths of P, from which S_P can be entirely reconstructed. In other words, the faces of a polyhedron define a collection of euclidean triangles that share edges as prescribed by K, and it is sufficient to parametrize the space of euclidean triangles for each of the triangles of $|K|$, to describe the possible intrinsic geometries of P.

2.1 Geometric realizations of a euclidean triangle

We parametrize the geometric realizations of a nondegenerate euclidean triangle in terms of its edgelengths and its angles as follows.

Definition 5. Let

$$g_\Delta\colon \mathbb{R}_+^3 \times (0,\pi)^3 \longrightarrow \mathbb{R}^3$$

be defined by

$$g_\Delta(a,b,c,\alpha,\beta,\delta) = \begin{pmatrix} a\cos\beta + b\cos\alpha - c \\ b\sin\alpha - a\sin\beta \\ \alpha+\beta+\gamma-\pi \end{pmatrix},$$

and set

$$\mathcal{T}_\Delta := g_\Delta^{-1}(0).$$

Proposition 6. *$\mathcal{T}_\Delta$ is an analytic submanifold of $\mathbb{R}_+^3 \times (0,\pi)^3$ of dimension 3, and $(a,b,c,\alpha,\beta,\gamma) \in \mathcal{T}_\Delta$ if and only if a, b and c are the edgelengths of a nondegenerate euclidean triangle with opposite angles α, β and γ, respectively.*

Proof. We compute

$$\mathrm{D}\,g_\Delta = \begin{pmatrix} \cos\beta & \cos\alpha & -1 & -b\sin\alpha & -a\sin\beta & 0 \\ -\sin\beta & \sin\alpha & 0 & b\cos\alpha & -a\cos\beta & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and find that, for example, columns 3, 2 and 6 form an upper-triangular 3×3 submatrix whose diagonal entries are nonzero on $\mathbb{R}_+^3 \times (0,\pi)^3$ and in particular on $\mathcal{T}_\Delta$. Hence, on $\mathcal{T}_\Delta$,

$$\mathrm{rk}\,\mathrm{D}\,g_\Delta = 3$$

on $\mathcal{T}_\Delta$. Therefore, by the Preimage Theorem, confer for instance [7], $\mathcal{T}_\Delta$ is an analytic submanifold of $\mathbb{R}_+^3 \times (0,\pi)^3$ of dimension

$$\dim \mathcal{T}_\Delta = \dim(\mathbb{R}_+^3 \times (0,\pi)^3) - \operatorname{rk} \mathrm{D}\, g_\Delta = 3.$$

Note that if a, b, c and α, β and γ are the edgelengths and opposite angles of a euclidean triangle then it is well known that they satisfy the claimed equations. The converse is an easy exercise in plane euclidean geometry. □

2.2 Geometric realizations of triangulated flat cone surfaces

Let K be a combinatoric and for every map $\ell \in \mathbb{R}_+^{K_1}$ and map $\sigma \in (0,\pi)^{C_K}$ abbreviate

$$\ell_{ij} := \ell(\{ij\}) \quad \text{and} \quad \sigma_{ijk} := \sigma\left((ijk)\right).$$

For every two-simplex $\{ijk\} \in K_2$ we parametrize a euclidean triangle with edgelengths ℓ_{ij}, ℓ_{jk} and ℓ_{ik} and opposite angles σ_{ikj}, σ_{jik} and σ_{ijk}, respectively, as follows.

Definition 7. Let

$$\widetilde{g}_K^{\text{in}} : \mathbb{R}_+^{K_1} \times (0,\pi)^{C_K} \longrightarrow (\mathbb{R}^3)^{K_2}$$

be the function defined by

$$\left(\widetilde{g}_K^{\text{in}}(\ell,\sigma)\right)(\{ijk\}) := g_\Delta(\ell_{ij}, \ell_{jk}, \ell_{ki}, \sigma_{ikj}, \sigma_{jik}, \sigma_{kji})$$

and let

$$\widetilde{\mathcal{T}}_K^{\text{in}} := \left(\widetilde{g}_K^{\text{in}}\right)^{-1}(0).$$

Remark 8. Note that $\left(\widetilde{g}_K^{\text{in}}\right)(\{ijk\})$ is neither symmetric in ℓ_{ij}, ℓ_{jk} and ℓ_{ki}, nor in σ_{ijk}, σ_{jki} and σ_{kij}. This means that strictly speaking $\widetilde{g}_K^{\text{in}}$ depends on the choice of a cyclic order of the zero-simplices of every two-simplex. However, any solution$(a,b,c,\alpha,\beta,\gamma) \in \mathcal{T}_\Delta$, also defines a solution $(b,c,a,\beta,\gamma,\alpha) \in \mathcal{T}_\Delta$. Hence, even though the function $\widetilde{g}_K^{\text{in}}$ depends on the choice of an order of the zero-simplices of every two-simplex of K, the set of solutions $\widetilde{\mathcal{T}}_K^{\text{in}}$ does not, and we omit the choice in the notation.

Proposition 9. *Let K be a combinatoric. Then $\widetilde{\mathcal{T}}_K^{in}$ is an analytic submanifold of $(0,\pi)^{C_K} \times \mathbb{R}_+^{K_1}$ of dimension E, and $(\ell,\sigma) \in \widetilde{\mathcal{T}}_K^{in}$ if and only if ℓ are the edgelengths of a flat cone surface triangulated by K and σ are its angles.*

Proof. Each two-simplex $\{ijk\} \in K_2$ contributes three rows to $\mathrm{D}\,\widetilde{g}_K^{\text{in}}$, which we denote by r_{ijk}, s_{ijk} and t_{ijk} and which, omitting the columns that are zero, look as follows

$$\begin{pmatrix} r_{ijk} \\ s_{ijk} \\ t_{ijk} \end{pmatrix} := \begin{pmatrix} -\ell_{ij}\sin\sigma_{ijk} & -\ell_{ik}\sin\sigma_{jki} & 0 & \cos\sigma_{ijk} & \cos\sigma_{jki} & -1 \\ \ell_{ij}\cos\sigma_{ijk} & -\ell_{ik}\cos\sigma_{jki} & 0 & \sin\sigma_{ijk} & -\sin\sigma_{jki} & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

where the first three columns shown correspond to the partial derivatives in the angles of $\{ijk\}$. Since $\tilde{g}_K^{\text{in}}$ only depends on the angles of $\{ijk\}$ on $\{ijk\}$, the other entries of the first three columns are zero. Suppose now for every $\{ijk\} \in K_2$ there were constants $\lambda_{ijk}, \mu_{ijk}, \nu_{ijk} \in \mathbb{R}$, such that

$$\sum \lambda_{ijk} r_{ijk} + \sum \mu_{ijk} s_{ijk} + \sum \nu_{ijk} t_{ijk} = 0. \tag{1}$$

Then, since each t_{ijk} contains an entry 1 in a column whose other entries are all 0, it follows that $\nu_{ijk} = 0$ for all $\{ijk\} \in K_2$. Together with 1 this implies

$$\sum \lambda_{ijk} r_{ijk} + \sum \mu_{ijk} s_{ijk} = 0. \tag{2}$$

Therefore, reading 2 for the first column means that

$$-\lambda_{ijk}\ell_{ij}\sin\sigma_{ijk} + \mu_{ijk}\ell_{ij}\cos\sigma_{ijk} = 0, \tag{3}$$

and reading it for the second column means that

$$-\lambda_{ijk}\ell_{ik}\sin\sigma_{jki} - \mu_{ijk}\ell_{ik}\cos\sigma_{jki} = 0. \tag{4}$$

Since ℓ_{ij} and ℓ_{ik} are nonzero, we can divide Equation 3 and Equation 4 by ℓ_{ij}, respectively ℓ_{ik}, and get

$$\mu_{ijk}\cos\sigma_{ijk} = \lambda_{ijk}\sin\sigma_{ijk}, \tag{5}$$

respectively

$$\lambda_{ijk}\sin\sigma_{jki} = -\mu_{ijk}\cos\sigma_{jki}. \tag{6}$$

If $\cos\sigma_{ijk} = 0$, then, since $\sigma_{ijk} \in (0,\pi)$ it follows that $\sigma_{ijk} = \frac{\pi}{2}$ and 5 becomes

$$0 = \lambda_{ijk}.$$

If $\cos\sigma_{ijk} \neq 0$, then we can divide Equation 5 by $\cos\sigma_{ijk}$ and get

$$\mu_{ijk} = \lambda_{ijk}\tan\sigma_{ijk}. \tag{7}$$

Similarly, we note that $\sin\sigma_{jki} \neq 0$ for $\sigma_{jki} \in (0,\pi)$, so that Equation 6 becomes

$$\lambda_{ijk} = -\mu_{ijk}\frac{1}{\tan\sigma_{jki}}. \tag{8}$$

Therefore, substitutingλ_{ijk} of Equation 8 into Equation 7, we get

$$\mu_{ijk} = -\mu_{ijk}\frac{\tan\sigma_{ijk}}{\tan\sigma_{jki}}. \tag{9}$$

If μ_{ijk} were nonzero, then we could divide Equation 9 by it and would get

$$\tan\sigma_{jki} = -\tan\sigma_{ijk}, \tag{10}$$

which for $\sigma_{ijk}, \sigma_{jki} \in (0, \pi)$ means

$$\pi - \sigma_{kji} = \sigma_{ijk}. \tag{11}$$

But, on $\widetilde{\mathcal{T}}_K^{\text{in}}$ we have that σ_{ijk}, σ_{jki} and σ_{kij} add up to π. Therefore, 11 implies that $\sigma_{kij} = 0$, which contradicts $\sigma_{kij} \in (0, \pi)$. It follows that $\mu_{ijk} = 0$.

Similarly, substituting μ_{ijk} of Equation 7 into Equation 8, we get

$$\lambda_{ijk} = -\lambda_{ijk}\frac{\tan\sigma_{ijk}}{\tan\sigma_{jki}}, \tag{12}$$

so that if λ_{ijk} were nonzero, then 12 would also imply 10, which again is impossible. It follows that $\lambda_{ijk} = 0$.

Hence, we have shown that on $\widetilde{\mathcal{T}}_K^{\text{in}}$ the rows of $\mathrm{D}\,\tilde{g}_K^{\text{in}}$ are linearly independent and that it is therefore of full rank. This proves that 0 is a regular value of $\tilde{g}_K^{\text{in}}$ so that by the preimage theorem, confer for instance [7], we get that $\widetilde{\mathcal{T}}_K^{\text{in}}$ is a smooth submanifold of dimension

$$\begin{aligned}\dim\widetilde{\mathcal{T}}_K^{\text{in}} &= \dim\left((0,\pi)^{C_K} \times \mathbb{R}_+^{K_1}\right) - \dim\left((\mathbb{R}^3)^{K_2}\right)\\ &= (3F + E) - 3F = E.\end{aligned}$$

The rest of the claim follows by construction of $\tilde{g}_K^{\text{in}}$ and Proposition 6. □

It is intuitively clear that each of the edgelengths of a flat-cone surface can be chosen independently, which we have shown algebraically in the previous proposition. Similarly, we show below that the set of angles of a triangulated flat-cone surface together with one edgelength determines all of its edgelengths. We note that this means in particular that there are more constraints on the angles of a flat-cone surface, than that they sum to 2π on each face individually. Indeed, since they share edges the angles of two faces are related by a sequence of laws of the sines for a path of faces between them. Below we use these constraints to construct a diffeomorphism that incorporates all the information needed to fully describe the set of angles of a triangulated flat-cone surface.

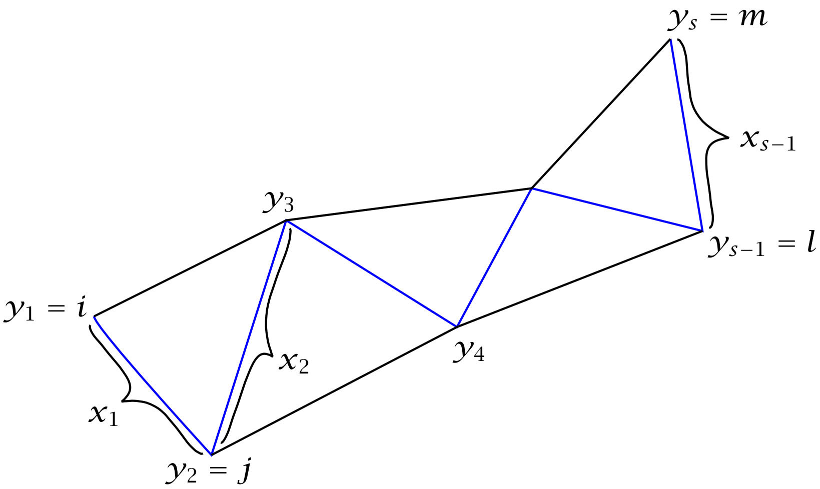


Figure 5: In blue: a path γ_{lm}.

Definition 10. Let p_Σ be the projection of $\mathbb{R}_+^{K_1} \times (0,\pi)^{C_K}$ onto $(0,\pi)^{C_K}$ and set

$$\mathcal{T}_K^{\text{in}} := p_\Sigma\left(\widetilde{\mathcal{T}}_K^{\text{in}}\right).$$

Proposition 11. *For every $\{ij\} \in K_1$ there is a diffeomorphism*

$$\varphi_{ij} \colon \mathcal{T}_K^{in} \times \mathbb{R}_+ \longrightarrow \widetilde{\mathcal{T}}_K^{in}.$$

In particular, $\mathcal{T}_K^{in}$ is a smooth submanifold of $(0,\pi)^{C_K}$ of dimension

$$\dim \mathcal{T}_K^{in} = E - 1.$$

Proof. We start with the following observation for a given $(\sigma, x) \in \widetilde{\mathcal{T}}_K^{\text{in}}$. For every $\{lm\} \in K_1 \setminus \{ij\}$ choose a path γ_{lm} of one-simplices from $\{ij\}$ to $\{lm\}$, that is, a finite sequence of consecutively adjacent one-simplices, whose first element is $\{ij\}$ and whose last element is $\{lm\}$. Denote this choice of paths by $\gamma := (\gamma_{lm})_{\{lm\}\in K_1\setminus\{ij\}}$. Denote now

$$y_1 = i, y_2 = j, y_3, \dots, y_{s-1} = l, y_s = m$$

the zero-simplices transversed by γ_{lm} from i to m. Note that $y_1, \dots,\ y_s$ depend on γ and on $\{lm\}$ but we omit this dependence in the notation for the sake of readability. Denote

$$x_k := x_{y_k y_{k+1}},$$

and compare with Figure 5. By Proposition 9 for every two-simplex $\{i'j'k'\} \in K_2$ the values $x_{i'j'}$, $x_{j'k'}$ and $x_{k'i'}$ are the edgelengths of a euclidean triangle with opposite angles $\sigma(j'k'i')$, $\sigma(k'i'j')$ and $\sigma(i'j'k')$. Therefore, they obey the law of the sines. In particular, we have,

$$x_{k+1} = \frac{\sin\sigma(y_{k+1}y_{k-1}y_{k+2})}{\sin\sigma(y_k y_{k+2} y_{k+1})} x_k,$$

for $k \in \{1, 2, \ldots, s-2\}$, so that

$$x_{lm} = x_{s-1} = \frac{\sin\sigma(y_{s-1}y_{s-2}y_s)}{\sin\sigma(y_{s-2}y_s y_{s-1})} \cdots \frac{\sin\sigma(y_2 y_1 y_3)}{\sin\sigma(y_1 y_3 y_2)} x_{ij}. \tag{13}$$

Now, note that $\sigma \in \mathcal{T}_K^{\text{in}}$ means that σ are the angles of a flat cone surface triangulated by K, and in particular, that there exists a flat cone surface $s \in \mathbb{R}_+^{K_1}$ triangulated by K, whose angles are σ. Note, furthermore, that if s is a flat cone surface triangulated by K with angles σ, then for every $\lambda \in \mathbb{R}_+$ the tuple $(\sigma, \lambda s)$ also satisfies the defining equations of $\widetilde{\mathcal{T}}_K^{\text{in}}$, so that we have that $(\sigma, \lambda s) \in \widetilde{\mathcal{T}}_K^{\text{in}}$. Therefore, for every $(\sigma, \ell) \in \mathcal{T}_K^{\text{in}} \times \mathbb{R}_+$ there is a flat cone surface triangulated by K, whose edgelength of $\{ij\}$ is ℓ. By our previous observation its other edgelengths are given by

$$d_{lm}^{\gamma}(\sigma, \ell) := \frac{\sin\sigma(y_{s-1}y_{s-2}y_s)}{\sin\sigma(y_{s-2}y_s y_{s-1})} \cdots \frac{\sin\sigma(y_2 y_1 y_3)}{\sin\sigma(y_1 y_3 y_2)} \ell, \tag{14}$$

for $\{lm\} \in K_1 \setminus \{ij\}$ and by ℓ for $\{ij\}$. We have now seen, that the map

$$\varphi_{ij}^{\gamma} \colon \mathcal{T}_K^{\text{in}} \times \mathbb{R}_+ \longrightarrow \widetilde{\mathcal{T}}_K^{\text{in}}$$

given by

$$\varphi_{ij}^{\gamma}(\sigma, \ell) := (\sigma, d^{\gamma}(\sigma, \ell)),$$

where $d^{\gamma}(\sigma, \ell)$ sends each one-simplex $\{lm\} \in K_1 \setminus \{ij\}$ to $d_{lm}^{\gamma}(\sigma, \ell)$ and $\{ij\}$ to ℓ, is well defined.

For $\sigma \in (0, \pi)^{C_K}$ the denominators in 14 do not vanish and therefore φ_{ij}^{γ} is smooth. Furthermore, we find that

$$(\varphi_{ij}^{\gamma})^{-1}(\sigma, x) := (p_{C_K}(\sigma, x), x_{ij})$$

defines a smooth inverse function for φ_{ij}^{γ} and that it is thus a diffeomorphism. The inverse function does not depend on the choice of γ. It follows that φ_{ij}^{γ} does not depend on the choice of γ and we can henceforth drop the reference to it. □

3 Parametrizing local extrinsic constraints

The intersection of a possibly rescaled polyhedron P with a unit sphere centered at a vertex is a spherical polygon whose edge-lengths are the angles of P around that vertex, and whose angles are the dihedral angles of P around that vertex. We use this observation to relate the surface and dihedral angles explicity in the following, by triangulating each of the spherical polygons thus defined, parametrizing the spherical triangles of the triangulation, and then eliminating the auxiliary variables.

3.1 Geometric realizations of a spherical triangle

We consider the unit sphere S^2 in $\mathbb{R}^3$ centered at the origin. For an introduction to spherical geometry see [2]. Recall that the intersection of S^2 with a plane passing through the origin is called a (spherical) *line*. A line on S^2 is a curve, so that it also makes sense to speak of line segments.

Definition 12. A *circuit* is a simplicial complex $Z = Z_0 \cup Z_1$ with zero-simplices

$$Z_0 = \{\{x_1\}, \dots, \{x_n\}\},$$

where $n > 2$ and with one-simplices

$$Z_1 = \{\{x_i x_{i+1}\} \mid i \in \{1, \dots, n-1\}\} \cup \{\{x_n x_1\}\}.$$

When we want to bring out the number n of zero-simplices of Z we also call Z an *n-circuit*.

Definition 13. A *spherical polygon* is a map $p\colon |Z| \to S^2$ from the geometric realization of a circuit Z into the unit sphere S^2, that maps one-simplices to line segments of lengths contained in $(0, \pi)$. If Z is an n-circuit, then we also call p a *spherical n-gon*.

We call the images of the zero-simplices of a spherical n-gon its *vertices* and the images of its one-simplices its *edges*. It is easy to see that a spherical 3-gon whose vertices define linearly independent vectors in $\mathbb{R}^3$ is injective. In particular, such a 3-gon is a simple closed curve on S^2 and by the Jordan Curve Theorem, divides the unit sphere into two regions.

Definition 14. The union of a spherical 3-gon whose vertices define linearly independent vectors in $\mathbb{R}^3$ with one of the two regions that it divides S^2 into is called a *spherical triangle*.

By definition a spherical triangle has edgelengths contained in $(0, \pi)$. Its *angle* at a vertex is the angle of the tangent vectors to the two adjacent edges that projects onto the interior of the spherical triangle. We easily find that the angles of a spherical triangle defined as above are either all contained in $(0, \pi)$, or all contained in $(\pi, 2\pi)$. This together with the spherical cosine rule motivates the following.

Definition 15. Let

$$g_3 \colon (0, \pi)^6 \cup \Big((0, \pi)^3 \times (\pi, 2\pi)^3\Big) \longrightarrow \mathbb{R}^3$$

be defined by

$$g_3(a, b, c, \alpha, \beta, \gamma) := \begin{pmatrix} \cos b \cos c + \sin b \sin c \cos \alpha - \cos a \\ \cos c \cos a + \sin c \sin a \cos \beta - \cos b \\ \cos a \cos b + \sin a \sin b \cos \gamma - \cos c \end{pmatrix},$$

and set

$$\mathcal{T}_3 := g_3^{-1}(0).$$

Proposition 16. *The set $\mathcal{T}_3$ is an analytic manifold of dimension 3 and $(a, b, c, \alpha, \beta, \gamma) \in \mathcal{T}_3$ if and only if there is a spherical triangle with edgelengths a, b and c and opposite angles α, β and γ, respectively.*

Proof. One easily finds a submatrix of $\mathrm{D}\, g_3$ which is of full rank on the domain of g_3. Indeed, any triple of partial derivatives by variables occuring together in a spherical congruence theorem, is of full rank 3. The first part of the claim then follows from the Preimage Theorem. The second part of the claim is an easy exercise in spherical geometry. □

3.2 Geometric realizations of a polyhedral cone in $\mathbb{R}^3$

Let K be a combinatoric, and $k \in K_0$ be a zero-simplex. Then we call the smallest simplical two-subcomplex of K that contains all the simplices of K that conain k, the *abstract cone* of k, and denote it by $C(k)$. The image of a simplex-wise linear and injective map $|C(k)| \longrightarrow \mathbb{R}^3$ defines a union of euclidean triangles in $\mathbb{R}^3$ with a common vertex, and the union of the planar sectors defined by them is called a *polyhedral cone*. In this way a polyhedron $P \colon |K| \longrightarrow \mathbb{R}^3$ defines a polyhedral cone $C(v)$ for each of its vertices v. If $k \in K_0$ has valency n_k, then we say that the polyhedral cone defined by P for k is an *n_k-cone*. We call v the vertex of $C(v)$, the half-lines at v defined by the edges of P, the *edges* of $C(v)$, and the plane sectors defined by the faces of P containing v, the *faces* of $C(v)$.

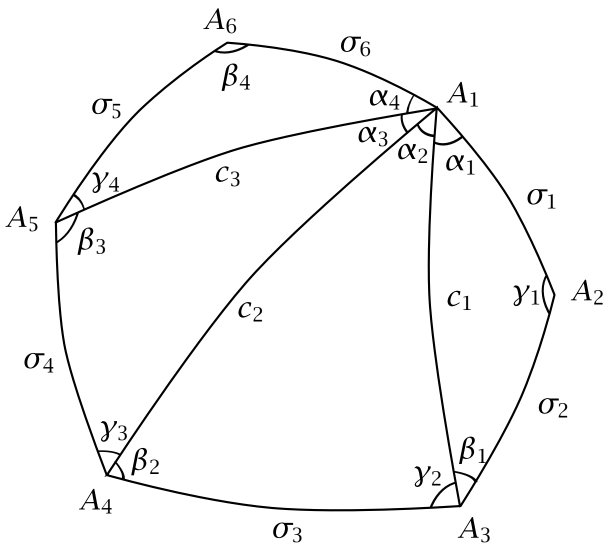


Figure 6: A triangulated spherical polygon

The intersection of a polyhedral cone $C(v)$ defined by a polyhedron P at its vertex v of valency n with a small unit sphere is a spherical n-gon, as defined earlier. To give it a triangulation we pick a cyclic labeling $A_1, A_2, \dots, A_n$ of its vertices and consider the spherical line-segments between A_1 and A_i for $i = 3, \dots, n-1$, compare Figure 6. Whenever no two edges of $C(v)$ are colinear then there is a unique shortest spherical line segment from A_1 to A_i and its length c_i is contained in $(0, \pi)$. Hence, in this situation the vertices A_1, A_i and A_{i+1} for $i = 2, \dots, n-2$ define a spherical three-gon Δ_{i-1}. If in addition no three edges of $C(v)$ are coplanar, then these spherical three-gons are injective and by Jordan's Curve Theorem define two disjoint regions of S^2 and the choice of any one of them makes Δ_i into a spherical triangle.

Definition 17. A polyhedral cone is called *in general position* if any triple of its edges is linearly independent.

Therefore, whenever $C(v)$ is in general position, we pick as interior of Δ_i the region of S^2 indicated by the inward pointing normal of P, and have defined a *triangulation* of $C(v)$. Let α_i, β_i, and γ_i be the angles of Δ_i, as shown in Figure 6.

Definition 18. Let

$$\tilde{g}_n \colon \Omega_3^{n-2} \times \mathbb{R}_+^{K_1} \longrightarrow \mathbb{R}^{3(n-2)+n}$$

be defined by

$$\widetilde{g}_n(\alpha,\beta,\gamma,c,\sigma,\delta) := \begin{pmatrix} g_3(\sigma_2,\sigma_1,c_1,\alpha_1,\beta_1,\gamma_1) \\ g_3(\sigma_3,c_1,c_2,\alpha_2,\beta_2,\gamma_2) \\ \vdots \\ g_3(\sigma_{n-2},c_{n-4},c_{n-3},\alpha_{n-3},\beta_{n-3},\gamma_{n-3}) \\ g_3(\sigma_{n-1},c_{n-3},\sigma_n,\alpha_{n-2}\beta_{n-2}\gamma_{n-2}) \\ \delta_1-(\alpha_1+\cdots+\alpha_{n-2}) \\ \delta_2-\gamma_1 \\ \delta_3-\beta_1-\gamma_2 \\ \vdots \\ \delta_{n-1}-\beta_{n-3}-\gamma_{n-2} \\ \delta_n-\beta_{n-2} \end{pmatrix}$$

and set

$$\widetilde{\mathcal{T}}_n := \widetilde{g}_n^{-1}(0) \quad \text{and} \quad \mathcal{T}_n := p_n(\widetilde{\mathcal{T}}_n),$$

where p_n is the projection onto the σ- and δ-factors of $\Omega_n := \Omega_3^{n-2}\times(0,2\pi)^{K_1}$.

Remark 19. To write $\Omega_3^{n-2}\times\mathbb{R}_+^{K_1}$, as we do above, as the domain for $\widetilde{g}_n$ is an abuse of notation, since $\widetilde{g}_n$ draws its variables from its domain as written in a different order. For a long clean ennummeration we refer the reader to the thesis version of this paper.

Proposition 20. *If P is a polyhedral cone in general position, then there is $\delta' \equiv \delta_P \mod 2\pi$ such that $(\sigma_P,\delta')\in\mathcal{T}_n$. Conversly, if $(\sigma,\delta)\in\mathcal{T}_n$, then there is a polyhedral cone P in general position, such that $\sigma_P=\sigma$ and $\delta_P\equiv\delta \mod 2\pi$.*

Proof. The first implication is true by construction. The converse holds by construction of $\mathcal{T}_n$ and by Proposition 16. □

Proposition 21. *$\widetilde{\mathcal{T}}_n$ is a real analytic submanifold of Ω_n of dimension $2n-3$.*

Proof. The function $\widetilde{g}_n$ is smooth. Denote r_1^i, r_2^i and r_3^i the rows of $\mathrm{D}\,\widetilde{g}_n$ corresponding to the i-th copy of $\mathrm{D}\,g_3$ in $D\widetilde{g}_n$ and $s_1,\dots,s_n$ the last n rows of $\mathrm{D}\,\widetilde{g}_n$. For $i\in\{1,\dots,n-2\}$, and $j\in\{1,\dots,n\}$ let λ_1^i, λ_2^i, λ_3^i and μ_j be elements in $\mathbb{R}$ such that

$$\sum_{i=1}^{n-2}\left(\lambda_1^i r_1^i+\lambda_2^i r_2^i+\lambda_3^i r_3^i\right)+\sum_{j=1}^{n}\mu_j s_j = 0. \tag{15}$$

Note that for every $j \in \{1, \ldots, n\}$ the column $\frac{\partial \tilde{g}_n}{\partial \delta_j}$ contains only one nonzero entry, which is in the row s_j and is equal to 1. Therefore, for every $j \in \{1, \ldots, n\}$ Equation 15 implies

$$\mu_j = 0 \tag{16}$$

Furthermore, on Ω_n the nonzero entries of the column $\frac{\partial \tilde{g}_n}{\partial \alpha_i}$ are in the rows r_1^i and s_1 for each $i \in \{1, \ldots, n-2\}$. Therefore, by Equation 15 and Equation 16, we get that for all $i \in \{1, \ldots, n-2\}$

$$\lambda_1^i = 0.$$

By the analogous argument for the columns $\frac{\partial \tilde{g}_n}{\partial \beta_i}$ and $\frac{\partial \tilde{g}_n}{\partial \gamma_i}$, we get for every $i \in \{1, \ldots, n-2\}$

$$\lambda_2^i = \lambda_3^i = 0.$$

Hence, $\mathrm{D}\,\tilde{g}_n$ is of full rank $3(n-2)+n$ on Ω_n and in particular 0 is a regular value of $\tilde{g}_n$. By the preimage theorem it follows that $\widetilde{\mathcal{T}}_n$ is a real analytic submanifold of Ω_n of dimension

$$\dim \widetilde{\mathcal{T}}_n = \dim \Omega_n - \operatorname{rk} \mathrm{D}\,\tilde{g}_n = 2n - 3.$$

□

Remark 22.

Proposition 23. *$\widetilde{\mathcal{T}}_n$ and $\mathcal{T}_n$ are diffeomorphic. Furthermore, $\mathcal{T}_n$ is real analytic.*

Proof. Note that the projection

$$\begin{aligned} p_n \colon \widetilde{\mathcal{T}}_n &\longrightarrow \mathcal{T}_n \\ (\sigma, \delta, \alpha, \beta, \gamma, c) &\longmapsto (\sigma, \delta) \end{aligned}$$

is smooth and surjective. It remains to construct a smooth inverse function. Let therefore $(\sigma, \delta) \in \mathcal{T}_n$. Then, by definition of $\mathcal{T}_n$ there are

$$(\alpha, \beta, \gamma, c) \in p(\Omega_n),$$

where p is the projection onto the corresponding factors of Ω_n and such that

$$\tilde{g}_n(\sigma, \delta, \alpha, \beta, \gamma, c) = 0. \tag{17}$$

We show that α, β, γ and c are smooth fuctions of σ and δ by induction on their components α_i, β_i, γ_i and c_i.

For $i = 1$ note that Equation 17 contains the equation

$$\cos c_1 = \cos \sigma_1 \cos \sigma_2 + \sin \sigma_1 \sin \sigma_2 \cos \delta_2. \tag{18}$$

Since the cosine function is invertible on $(0, \pi)$ Equation 18 implies that c_1 is a smooth function $c_1(\sigma, \delta)$ of σ and δ. Furthermore, note that by Equation 17 $\gamma_1 = \delta_2$ is a smooth function $\gamma_1(\sigma, \delta)$ of σ and δ. In addition Equation 17 contains the equation

$$\cos \sigma_2 = \cos \sigma_1 \cos c_1 + \sin \sigma_1 \sin c_1 \cos \alpha_1. \tag{19}$$

Since $\delta_2 = \gamma_1$ is given, we know whether $(\alpha_1, \beta_1, \gamma_1) \in (0, \pi)^3$ or whether $(\alpha_1, \beta_1, \gamma_1) \in (\pi, 2\pi)^3$. The cosine function is invertible both on $(0, \pi)$ and on $(\pi, 2\pi)$. Therefore, since $c_1 = c_1(\sigma, \delta)$ is a smooth function of σ and δ Equation 19 implies that α_1 is a smooth function $\alpha_1(\sigma, \delta)$ of σ and δ. By an analogous argument β_1 is a smooth function $\beta_1(\sigma, \delta)$ of σ and δ.

Suppose now that α_i, β_i, γ_i and c_i are smooth functions of σ and δ for $i \in \{1, \ldots, m-1\}$ and $1 \leq m \leq n-2$. Then Equation 17 contains the equation $\gamma_m = \delta_{m+1} - \beta_{m-1}$, which defines by induction hypothesis γ_m as a smooth function $\gamma_m(\sigma, \delta)$ of σ and δ. Furthermore, by an argument identical to the argument for the case $m = 1$ but with shifted indices, we find that c_m, α_m and β_m are smooth functions $\alpha_m(\sigma, \delta)$, $\alpha_m(\sigma, \delta)$ and $\beta_m(\sigma, \delta)$ of σ and δ.

Hence, we have shown, that α, β, γ and c are smooth functions $\alpha(\sigma, \delta)$, $\beta(\sigma, \delta)$, $\gamma(\sigma, \delta)$ and $c(\sigma, \delta)$ of σ and δ. Define

$$\begin{aligned} \psi_n \colon \mathcal{T}_n &\longrightarrow \widetilde{\mathcal{T}}_n \\ (\sigma, \delta) &\longmapsto (\sigma, \delta, \alpha(\sigma, \delta), \beta(\sigma, \delta), \gamma(\sigma, \delta), c(\sigma, \delta)). \end{aligned}$$

Then ψ_n is a smooth inverse function of p_n, so that ψ_n and p_n are diffeomorphisms. □

3.3 The collection of polyhedral cones of a polyhedron

Let K be a combinatoric and $k \in K_0$ be a zero-simplex. We note that a triangulation of a polyhedral cone realizing $C(k)$, can be determined independent of the geometric realization. Let τ therefore be a choice of a triangulation for every zero-simplex $k \in K_0$. Then we denote $\alpha^k, \beta^k, \delta^k, c^k, \sigma^k$ and δ^k the variables for angles $\alpha_i^k, \beta_i^k, \delta_i^k$ of the spherical triangles given by τ at k, c_i^k be the variables for lengths of the diagonals of τ at k, and σ_i^k and δ_i^k be the variables for the edgelengths and angles of the spherical n_k-gon to be parametrized. Let furthermore

$$\Omega_{k,\tau} := (\Omega_{n_k})^{K_0},$$

and denote $\alpha, \beta, \gamma, c, \sigma$ and δ the vectors containing $\alpha^k, \beta^k, \gamma^k, c^k \sigma^k$ and δ^k for every zero-simplex $k \in K_0$. Then, we can define with slight abuse of notation the following.

Definition 24. Let K be a combinatoric and τ be a choice of triangulation for every $k \in K_0$. Let

$$\widetilde{g}^{\mathrm{le}}_{K,\tau} \colon \Omega_{K,\tau} \longrightarrow \bigtimes_{k \in K_0} \mathbb{R}^{3(n_k-2)+n_k}$$

be defined by

$$\widetilde{g}^{\mathrm{le}}_{K,\tau}(\sigma, \delta, \alpha, \beta, \gamma, c)(\{k\}) := \widetilde{g}_{n_k}\left(\sigma^k, \delta^k, \alpha^k, \beta^k, \gamma^k, c^k\right),$$

and set

$$\widetilde{\mathcal{T}}^{\mathrm{le}}_{K,\tau} := \left(\widetilde{g}^{\mathrm{le}}_{K,\tau}\right)^{-1}(0).$$

Proposition 25. *Let K be a combinatoric and τ be a triangulation of K_0. Let g be the genus of the surface $|K|$. Then,*

$$(\sigma, \delta, \alpha, \beta, \gamma, c) \in \widetilde{\mathcal{T}}^{le}_{K,\tau} \Leftrightarrow \forall k \in K_0 \ \ (\sigma^k, \delta^k, \alpha^k, \beta^k, \gamma^k, c^k) \in \widetilde{\mathcal{T}}_{n_k}.$$

Furthermore, $\widetilde{\mathcal{T}}^{le}_{K,\tau}$ is a smooth submanifold of $\Omega_{K,\tau}$ of dimension

$$\dim \widetilde{\mathcal{T}}^{le}_{K,\tau} = 3F + 6g - 6 = 2E + 6g - 6.$$

To prove this proposition we will eliminate the linearized system of equations $\mathrm{D}\,\widetilde{g}^{\mathrm{le}}_{K,\tau} = 0$. The major challenge in doing so will be to decide on an order in which we eliminate the variables and which equation we use at what moment. In terms of the matrix $\mathrm{D}\,\widetilde{g}^{\mathrm{le}}_{K,\tau}$ that means finding a good ordering of its columns and rows. Once a good ordering is found the proof will follow from the structure of the reordered matrix $\mathrm{D}\,\widetilde{g}^{\mathrm{le}}_{K,\tau}$. Note that $\mathrm{D}\,\widetilde{g}^{\mathrm{le}}_{K,\tau}$ consists of copies of $\mathrm{D}\,\widetilde{g}_n$, for different $n = n_k$ depending on $k \in K_0$, and recall from Remark 22 that we can eliminate $\mathrm{D}\,\widetilde{g}_n$ in terms of any subset of cardinality $2n - 3$ of the partial derivatives by surface and by dihedral angles. Putting it differently, we can determine all auxiliary variables and any set of three of the set of dihedral angles and surface angles in the linear system $\mathrm{D}\,\widetilde{g}_n = 0$ in terms of the remaining surface and dihedral angles. Hence, we can eliminate $\mathrm{D}\,\widetilde{g}^{\mathrm{le}}_{K,\tau}$ by eliminating for every $k \in K_0$ three of the variables σ^k and δ^k and all auxiliary variables α^k, β^k, γ^k and c^k. However, since every dihedral angle occurs as the dihedral angle of two polyhedral cones, and we want to eliminate every copy of $\mathrm{D}\,\widetilde{g}_{n_k}$ separately from the others, we need to make sure that no δ_{ij} gets chosen to be eliminated twice. Equivalently, to eliminate $\mathrm{D}\,\widetilde{g}^{\mathrm{le}}_{K,\tau}$ we need to choose for every $k \in K_0$ three elements of the set

$$\{\text{ One-simplices of } K \text{ incident in } k \ \} \cup \{\text{ Corners of } K \text{ at } k \ \},$$

in such a way that our choices for each $k \in K_0$ are disjoint. To do so we will dualize K below and we will make our "choices" by giving "colors" to the dual object K^*.

Definition 26. Let K be a combinatoric. The *dual graph* of K is the abstract simplicial one-complex K^*, such that

(i) there is a bijection $*\colon K_2 \longrightarrow K_0^*$, and

(ii) two zero-simplices of K^* are adjacent in K^* if and only if their preimages under $*$ are adjacent in K as two-simplices of K.

We denote $\{ijk\}^* := *(\{ijk\})$ and for $k \in K_0$ we call the n_k-circuit $C(k)^*$, by severe abuse of concepts, a *two-simplex of* K^* and denote it by k^*. We denote the set of two-simplices of K^* by K_2^*.

Remark 27. The dual graph K^* of a combinatoric K is well defined up to adjacency preserving bijections.

Definition 28. Let K be a combinatoric. A *path of two-simplices* of K^* is an ordered set of two-simplices of K^* such that any two consecutive elements contain a common one-simplex.

Remark 29. Let K be a combinatoric. Every one-simplex of K^* is of the form

$$\{\{ijk\}^*, \{jkl\}^*\}$$

and is contained in exactly two two-simplices, namely j^* and k^* of K^*, which we henceforth call *adjacent*. Every two-simplex of K^* contains at least three zero-simplices of K^*. Furthermore, K^* is a trivalent connected graph and for any two one-simplices s_1 and s_2 of K^* there is a path γ of two-simplices of K^* connecting them, that is, such that the first element of γ contains s_1 and the last element of γ contains s_2.

Definition 30. Let K be a combinatoric and k_1^* and k_2^* be two two-simplices of K^* that are adjacent in a one-simplex s . Then an *arrow* on s is a bijective map $a \in \{k_1, k_2\}^{\{0,1\}}$. We say that *a points inward to* $a(1)$ and that it *points outward of* $a(0)$.

Definition 31. Let K be a combinatoric. Then, the set of *corners* of K^* is defined by

$$C_{K^*} := \left\{ (ijk) \,\middle|\, \begin{array}{c} (ijk) \text{ is an order of } \{i,j,k\} \subset K_0^*, \\ \text{where } \{ij\}, \{jk\} \in K_1^* \end{array} \right\} \Big/ \sim,$$

where $\sim$ identifies an order (ijk) with the order (ijk). We say that $(ijk) \in C_{K^*}$ is a *corner of* K^* *at* j. Furthermore, for a two-simplex k^* of K^*, we call the set of corners (ijl) of K^* where $\{ij\}, \{jl\} \in k^*$ the *corners of* k^*.

Definition 32. Let K be a combinatoric. A *coloring* of K^* is a subset $C \subset C_{K^*}$ together with a set A of arrows on the one-simplices of K^*. We call the elements of $C \cup A$ the *colors*, call the corners contained in C, and the one-simplices contained in A *colored*, and say that a one-simplex of K^* is *colored by* the arrows on it in A.

Definition 33. A coloring (C, A) of K^* is called *admissible* if for every two-simplex k^* of K^* it contains exactly three elements of the following

$$\{\text{Arrows pointing outward of } k^*\} \cup \{\text{Corners of } k^*\}.$$

Starting with any specific combinatoric K one can easily find many different admissible colorings of K^*. It is also easy to see that there always is an admissible coloring. For instance, since every two-simplex of K^* contains at least three zero-simplices, one can pick three corners of every two-simplex of K^*. For the purpose of proving Proposition 25 this kind of coloring is already sufficient. However, to prove Theorem 42 we will want an admissible coloring that colors as many one-simplices of K^* by exactly one arrow, which is what following conjecture would achieve.

Conjecture 34 (Elimination Pattern). *Let K be a combinatoric, and g be the genus of $|K|$. Then there is an admissible coloring of K^* such that*

(i) If $g = 0$, then there is exactly one arrow on every one-simplex of K^* and exactly six corners of K^* are colored. Furthermore, for any two adjacent zero-simplices i and j of K^* the colored corners can be chosen as the union of the corners of i and the corners of j.

(ii) If $g \geq 1$, then there is exactly one arrow on all except $6g - 6$ of the one-simplices of K^* and no corners are colored.

Proof of Proposition 25 modulo the Elimination Pattern Conjecture. Let

$$l\colon \{1, \dots, V\} \to K_0$$

be a bijective map and (C, A) be an admissible coloring of K^* as obtained by the conclusion of the Elimimation Pattern Conjecture 34. For $k \in K_0$ denote C_k the set of colored corners of k^*, and A_k the set of one-simplices of k^* that is colored with an outward pointing arrow. Let C_n be the set of columns of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ that consists of the partial derivatives

$$\frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \alpha^k}, \frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \beta^k}, \frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \gamma^k}, \frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial c^k}, \frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \sigma_{ikj}}, \text{ and } \frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \delta_{ik}}$$

for $k = l(n)$, all $(ikj)^* \in C_k$ and all $\{jk\}^* \in A_k$. Denote $\mathcal{R}_n$ the set of rows of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ of the components $\tilde{g}^{\mathrm{le}}_{K,\tau}(\{l(n)\})$ of $\tilde{g}^{\mathrm{le}}_{K,\tau}$. Denote

$$C_{\mathrm{rest}} := \{\frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \sigma}, \frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \delta}\} \setminus \cup_{\ell=1}^{V} C_\ell.$$

Choose an ordering of the columns of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ such that

$$\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau} = (C_1 \mid \ldots \mid C_V \mid C_{\mathrm{rest}})$$

and such that for $\ell \in \{1, \ldots, V\}$ the columns

$$\mathcal{D}_\ell := \{\frac{\partial \tilde{g}^{\mathrm{le}}_{K,\tau}}{\partial \delta}\} \cap C_\ell$$

appear after the other columns in C_ℓ in $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$. Rearrange the rows of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ as follows

$$\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau} = \begin{pmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_V \end{pmatrix}.$$

With the chosen ordering of columns and rows of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ we henceforth consider C_ℓ and $\mathcal{R}_\ell$ as submatrices of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$.

Then, by the definition of $\tilde{g}^{\mathrm{le}}_{K,\tau}$ the submatrix $\mathcal{R}_1$ contains only non-zero entries in the columns that are the partial derivatives by the variables in the vectors α^k, β^k, γ^k, c^k, σ^k and δ^k, where $k = l(1)$ and their intersection with $\mathcal{R}_1$ is $\mathrm{D}\,\tilde{g}_n$, where $n = n_k$ is the valency of k. Hence, by Remark 22 we can fully eliminate $\mathcal{R}_1$ in such a way that the block $\mathcal{R}_1 \cap C_1$ becomes upper triangular. Denote C_1', respectively $\mathcal{D}_1'$, the columns obtained from C_1, respectively $\mathcal{D}_1'$ by this operation and likewise $\mathcal{R}_1'$ the rows obtained from $\mathcal{R}_1$. Then $C_1' \setminus \mathcal{D}_1'$ is upper triangular. Furthermore, $\mathcal{D}_1'$ contains a quadratic upper triangular submatrix with the same number of columns as $\mathcal{D}_1$ and that is contained in rows of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ that contain otherwise only zeros. Hence, the other nonzero entries of $\mathcal{D}_1'$ can be eliminated using row operations of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ without changing $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau} \setminus C_1'$. Then the submatrix C_1'' obtained from C_1' by this operation is upper triangular. We successively repeat the argument for $C_2, \ldots, C_V$ and thereby fully eliminate $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$.

We compute the rank of $\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau}$ by counting its independent columns $\cup_{\ell=1}^{V} C_\ell$ as follows

$$\mathrm{rk}\,\mathrm{D}\,\tilde{g}^{\mathrm{le}}_{K,\tau} = \sum_{k \in K_0} \big(3(n_k - 2) + (n_k - 3)\big) + 3V,$$

where the first term counts the number partial derivatives in the variables α^k, β^k, γ^k and c^k and the second term counts the cardinality of $\cup_{\ell=1}^{V}\mathcal{D}_\ell$, which is the same as the total number of colors of (C,A). Similarly, we find

$$\dim \Omega_{K,\tau}^{\mathrm{le}} = E + 3F + \sum_{k\in K_0} \left(3(n_k - 2) + (n_k - 3)\right),$$

and thus by the Preimage Theorem, confer e.g. [7], we compute

$$\begin{aligned}\dim \widetilde{\mathcal{T}}_{K,\tau}^{\mathrm{le}} &= \dim \Omega_{K,\tau}^{\mathrm{le}} - \operatorname{rk} \mathrm{D}\,\widetilde{g}_{K,\tau}^{\mathrm{le}} &(20)\\ &= 3F + E - 3V &(21)\\ &= 3F + 6g - 6 &(22)\end{aligned}$$

where the last equation holds by Euler's Polyhedron Formula applied to V and because in any abstract simplicial two-complex K we have that $3F = 2E$. □

4 Wrapping it up

In this section we reconstruct a simply connected polyhedron from its polyhedral cones and its faces. We define the moduli-space $\widetilde{\mathcal{T}}_{K,\tau}$ of geometric realizations up to similarities of $\mathbb{R}^3$ of a simply connected polyhedron with combinatoric K correspondingly. We recompute its dimension depending on the combinatorc K modulo the Elimination Pattern Conjecture 34, reparametrize $\widetilde{\mathcal{T}}_{K,\tau}$ as $\mathcal{T}_K$ in terms of dihedral and face angles only and finally draw our first geomtric conclusion about the set of flexible polyhedra.

4.1 Geometric realizations of simply connected polyhedra

To parametrize a simply connected polyhedron we parametrize its collection of polyhedral cones using $\widetilde{g}_{K,\tau}^{\mathrm{le}}$ and its collection of faces using $\widetilde{g}_K^{\mathrm{in}}$. Below we show that this is sufficient.

Definition 35. Define

$$\widetilde{g}_{K,\tau} \colon \Omega_K \times \mathbb{R}_+^{K_1} \longrightarrow \left(\mathbb{R}^3\right)^{K_2} \times_{k\in K_0} \mathbb{R}^{3(n_k-2)+n_k}$$

by

$$\widetilde{g}_{K,\tau}(\sigma,\delta,\alpha,\beta,\gamma,c,x)(\{k\}) := \begin{pmatrix} \widetilde{g}_K^{\mathrm{in}}(\sigma,x) \\ \widetilde{g}_{K,\tau}^{\mathrm{le}}(\sigma,\delta,\alpha,\beta,\gamma,c)(\{k\}) \end{pmatrix},$$

and set

$$\widetilde{\mathcal{T}}_{K,\tau} := \left(\widetilde{g}_{K,\tau}\right)^{-1}(0) \quad \text{and} \quad \mathcal{T}_K := p(\widetilde{\mathcal{T}}_{K,\tau}),$$

where p is the projection onto the σ- and δ-factors.

Remark 36. Note that, by definition $\widetilde{\mathcal{T}}_{K,\tau} = \widetilde{\mathcal{T}}^{\mathrm{le}}_{K,\tau} \cap \widetilde{\mathcal{T}}^{\mathrm{in}}_{K}$.

37 Main Theorem. *Let $P\colon |K| \to \mathbb{R}^3$ be a polyhedron that is locally in general position, then there is $\delta' \equiv \delta_P \mod 2\pi$ such that $(\sigma_P, \delta') \in \mathcal{T}_K$. If K is a combinatoric such that $|K|$ is simply connected then the converse holds, i.e., if $(\sigma, \delta) \in \mathcal{T}_K$, then there is a polyhedron $P\colon |K| \to \mathbb{R}^3$ such that $\sigma_P = \sigma$ and $\delta_P \equiv \delta \mod 2\pi$.*

To construct a polyhedron $P\colon |K| \to \mathbb{R}^3$ with given surface angles σ and dihedral angles δ with $(\sigma, \delta) \in \mathcal{T}_K$ we will find a nice sequence of zero-simplices $k_1, \dots, k_n \in K_0$ and construct the polyhedral pieces P'_i that will be the faces of P around its vertex $P(|k_i|)$. At the same time we will successively fit them together in a sequence of polyhedral pieces

$$P_{i+1} := P'_i \cup P_i,$$

where by abuse of notation we have identified all maps with their image in $\mathbb{R}^3$. We stop the process after finitely many steps at $P = P_N$ for some $N \in \mathbb{N}$.

To ensure that we can take the union of P'_i and P_i in such a way that they define a well defined map P_{i+1}, we need to choose the sequence $k_1, \dots k_n \in K_0$ in such a way that the union of the cones

$$D(k_1, \dots, k_n) := \bigcup_{i=1}^{n} |C(k_i)|$$

has no handles. We achieve this by means of the following two lemmas.

Lemma 38. *Let K be a combinatoric such that $|K|$ is simply connected and $k_1, \dots, k_n \in K_0$ be zero-simplices such that $D(k_1, \dots, k_n)$ is a disc and $|K| \setminus D(k_1, \dots, k_n) \neq \emptyset$. Then there is a zero-simplex $k_{n+1} \in \partial D(k_1, \dots, k_n)$ such that $D(k_1, \dots k_{n+1})$ has one connected component and is contractible.*

Lemma 39. *Let D be a disc and $S = \partial D$ be marked with points $x_1, \dots, x_n \in S$. Then, there is no collection Γ of paths in D starting and ending in $\{x_1, \dots, x_n\}$ such that*

(i) Every marked point is starting or ending point for some path in Γ and for any two distinct points $x_i, x_j \in \{x_1, \dots, x_n\}$ there is at most one path $\gamma \in \Gamma$ starting in x_i and ending in x_j or vice versa.

(ii) No two paths of Γ cross.

(iii) Every path $\gamma \in \Gamma$ starts and ends in distinct points.

(iv) *No path* $\gamma \in \Gamma$ *starts and ends in* $x_i, x_j \in \{x_1, \dots, x_n\}$ *such that* x_i *and* x_j *are adjacent in* S.

Proof. We prove this lemma by induction on n. For $n = 1$ any collection of paths in D starting and ending in $\{x_1\}$ violates condition (3). Suppose now the statement of the lemma is true for any disc with $m \in \{1, \dots, n\}$ marked points on its boundary. Let D be a disc with $n+1$ marked points $x_1, \dots, x_{n+1} \in S = \partial D$. Let Γ be a collection of paths in D starting and ending in $\{x_1, \dots, x_{n+1}\}$ that satisfies conditions (1)-(3). We show that then Γ violates condition (4). Let $\gamma \in \Gamma$. Without loss of generality it starts in x_1 and ends in x_i for $i \in \{2, \dots, n+1\}$. The union S' of γ with the segments of S from x_1 to x_2, from x_2 to x_3, ..., and from x_{i-1} to x_i is a simple closed curve and by the Jordan Curve Theorem bound a disc $D' \subset D$. Then $S' = \partial D'$ is marked by $i < n+1$ points $x_1, \dots, x_i$. Let $\Gamma' \subset \Gamma$ be the subset of paths that start or end in $\{x_1, \dots, x_i\}$. If $i = 2$, then γ starts in x_1 and ends in x_2 that are adjacent so that Γ violates condition (4). If $i \neq 2$, then by condition (1) there are paths of Γ starting in $\{x_2, \dots, x_i\}$ and therefore $\Gamma' \neq \emptyset$. Since no two paths of Γ cross by condition (2) every path that starts or ends in D' does not cross γ and therefore by the Jordan Curve Theorem stays contained in D'. It follows that the paths in Γ' all start and end in $\{x_1, \dots, x_n\}$ and that we can apply the induction hypothesis to Γ', D' and S'. Since Γ' satisfies conditions (1)-(3) as a subset of Γ it follows that Γ' violates condition (4). This means that there is either a path $\gamma' \in \Gamma'$ that starts and ends in $\{x_1, x_i\}$, or that starts and ends in $\{x_j, x_{j+1}\}$ for $j \in \{2, i-1\}$. The first case implies that γ and γ' are two distinct paths in Γ from x_1 to x_i, which is impossible because Γ satisfies condition (1). In the second case γ' is a path in Γ between two adjacent markings of S so that Γ violates condition (4). □

Proof of Lemma 38. Since $D(k_1, \dots, k_n)$ is a closed disc in the sphere $|K|$, its complement

$$D := |K| \setminus D(k_1, \dots, k_n)$$

is an open disc. Its boundary $S := \partial D$ is an edgepath of $|K|$ whose vertices mark finitely many points on S. If the lemma is not true we will find below in the remainder of the proof for every vertex $|k|$ of S an edge that starts in $|k|$ and ends in a vertex $|i|$ of S that is not adjacent to k in S. Since edges are paths that do not cross each other this is impossible by Lemma 39.

First note that, for every vertex $|k| \subset S$,

$$|k| \in D(k_1, \dots, k_n) \cap |C(k)|.$$

Since $D(k_1, \dots, k_n)$ and $|C(k)|$ are connected, it follows that their union $D(k_1, \dots, k_n, k)$ is connected. Therefore, if the statement of the lemma is not true, then for every vertex $|k| \subset S$ there is a closed path γ_k that is non-contractible in $D(k_1,\ldots,k_n,k)$. Furthermore, γ_k is homotopic in $D(k_1,\ldots,k_n,k)$ to a closed simple non-contractible edgepath, which we also denote by γ_k. The intersection $D \cap \gamma_k$ is nonempty, since otherwise $\gamma_k \subset D(k_1,\ldots,k_n)$, and since $D(k_1,\ldots,k_n)$ is simply connected by hypothesis, γ_k would be contractible in $D(k_1,\ldots,k_n,k)$. Furthermore, since γ_k is a simple edgepath its connected components in $\overline{D}$ are edgepaths starting and ending at pairwise distinct vertices of S.

Denote by γ_k^{ij} the connected component of γ_k in $\overline{D}$ starting in the vertex $|i|$ of S and ending in the vertex $|j|$ of S. There is a connected component γ_k^{ij} of γ_k in $\overline{D}$ such that either $|i|$ or $|j|$ are not adjacent to $|k|$ in S; for otherwise, since $\gamma_k^{ij} \subset |C(k)|$ and $|C(k)|$ is simply connected, γ_k^{ij} would be homotopic in $|C(k)|$ to $|ik| \cup |kj|$ and γ_k contractible in $D(k_1,\ldots,k_n,k)$.

Since every connected component γ_k^{ij} of γ_k in $\overline{D}$ is a subset of $|C(k)|$ and $|C(k)|$ is simply connected there is furthermore a path $\tilde{\gamma}_k^{ij}$ that is homotopic to γ_k^{ij} in $|C(k)|$ and that contains $|k|$. Let $\tilde{\gamma}_k$ be the path homotopic to γ_k in $D(k_1,\ldots,k_n,k)$ obtained by replacing γ_k^{ij} by $\tilde{\gamma}_k^{ij}$.

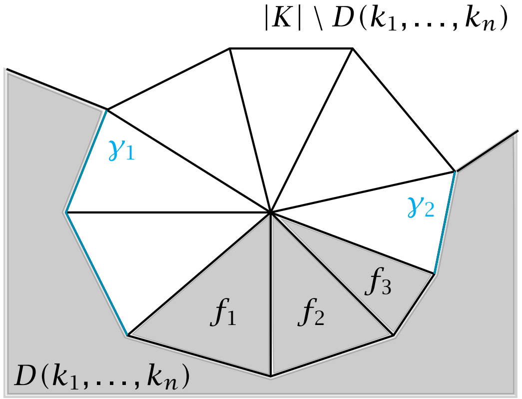
Consider the connected components γ_k^{ij} of γ_k in $\overline{D}$ such that $|i|$ or $|j|$ is not adjacent to $|k|$ in S. Without loss of generality it is $|i|$ that is not adjacent to $|k|$ in S. Consider the segments $\overline{\gamma}_k^{ij}$ of $\tilde{\gamma}_k^{ij}$ starting in $|k|$ and ending in $|i|$. Then, since $i \in \operatorname{st} k$ there is an edge $|ik| \subset |C(k)|$ and since both $|ik|$ and $\overline{\gamma}_k^{ij}$ are paths in $|C(k)|$ that start in $|k|$ and end in $|i|$ they are homotopic in $|C(k)|$.

Let $\overline{\gamma_k}$ be the path obtained from $\widetilde{\gamma_k}$ by replacing the segments $\overline{\gamma}_k^{ij}$ of $\tilde{\gamma}_k^{ij}$ by the edges $|ik|$ whenever $|i|$ is not adjacent to $|k|$ in S. If for all connected components γ_k^{ij} of γ_k for which $|i|$ is not adjacent to $|k|$ in S, the edge $|ik|$ was contained in $D(k_1,\ldots,k_n)$ already, then $\overline{\gamma}_k$ would be contractible in $D(k_1,\ldots,k_n,k)$. But γ_k is homotopic to $\overline{\gamma}_k$ in $D(k_1,\ldots,k_n) \cup |C(k)|$ and therefore γ_k would be contractible in $D(k_1,\ldots,k_n) \cup |C(k)|$. It follows that, for every vertex $|k|$ of S there is an edge $|ik|$ in $\overline{D}$ and such that $|i|$ is not adjacent to $|k|$ in S. □

Lemma 40. *Let K be a combinatoric such that $|K|$ is simply connected and let $k_1,\ldots,k_N \in K_0$ be zero-simplices of K such that for every $n \in \{1,\ldots,N-1\}$ the set $D(k_1,\ldots,k_n)$ satisfies the hypothesis of the previous Lemma 38, and $k_{n+1} \in K_0$ satisfies the conclusions of the previous Lemma 38. Then,*

$$|C(k_{n+1})| \cap D(k_1,\ldots,k_n) = \cup_{l=1}^{m} f_l \cup \gamma_1 \cup \gamma_2,$$

where $f_1,\ldots,f_m$ are consecutively adjacent faces of $|K|$ with the common vertex $|k_{n+1}|$ and γ_1 and γ_2 are two edge-paths of $|K|$ that start at a vertex of f_1, *respectively f_m, not contained in $f_2, \dots, f_{m-1}$. That is, the above intersection looks e.g. as follows,*



where the depicted cone is $|C(k_{n+1})|$, the shaded region is $D(k_1, \dots, k_n)$ and the unshaded region is its complement in $|K|$.

Proof. We first note that $D(k_1) = |C(k_1)|$ is such that for every zero-simplex $|s| \in \partial D(k_1)$ in its boundary, the faces it shares with $|C(s)|$ are two adjacent faces, and that $D(k_1) \setminus |C(s)|$ is edge-path-connected and $|C(s)| \cap D(k_1)$ satisfies the conclusion of the lemma. Now suppose that for some $n \in \{1, \dots, N-1\}$ the set $D(k_1, \dots, k_n)$ is such that for every zero-simplex $s \in \partial D(k_1, \dots, k_n)$ the faces it shares with $|C(s)|$ are consecutively adjacent, and that the closure of $D(k_1, \dots, k_n) \setminus |C(s)|$ is edge-path-connected.

Then, if $x_1, x_2 \in \partial|C(k_{n+1})| \cap D(k_1, \dots, k_n)$ are two zero-simplices, there is a simple edge-path γ' in the closure of $D(k_1, \dots, k_n) \setminus |C(k_{n+1})|$ from x_1 to x_2. Let β_1 and β_2 be the two distinct edgepaths from x_1 to x_2 on $\partial|C(k_{n+1})|$. Then $\gamma := \gamma' \cup \beta_1$ is a simple closed path and by the Jordan Curve Theorem bounds two disjoint open discs D_1 and D_2 on $|K|$ such that $D_1 \cup \gamma \cup D_2 = |K|$. Since β_2 intersects γ only in its starting and ending points, its interior is contained in either D_1 or in D_2. Let without loss of generality D_2 be the disc containing the interior of β_2. Then D_2 also contains the interior of $|C(k_{n+1})|$. It follows, that $D_1 \cap |C(k_{n+1})| = \emptyset$.

Since by the previous Lemma 38 the set $D(k_1, \dots, k_{n+1})$ is contractible, it follows that D_1 or D_2 is contained in $D(k_1, \dots, k_{n+1})$. If $D_1 \subset D(k_1, \dots, k_{n+1})$, then, since $D_1 \cap |C(k_{n+1})| = \emptyset$ it follows that $D_1 \subset D(k_1, \dots, k_n)$ and therefore the closure $D_1 \cup \gamma$ of D_1 is also contained in $D(k_1, \dots, k_n)$. Since $\beta_1 \subset \gamma$ we have that $\beta_1 \subset D(k_1, \dots, k_n)$. If D_1 is not contained in $D(k_1, \dots, k_{n+1})$, then D_2 is contained in $D(k_1, \dots, k_{n+1})$. However, then $|C(k_{n+1})|$ is a homotopy in D_2 from β_1 to β_2 and therefore D_2 can be retracted to the complement of the interior of $|C(k_{n+1})|$. Hence , $\gamma' \cup \beta_2$ bounds a disc in $D(k_1, \dots, k_n)$ and in particular we have that $\beta_2 \subset D(k_1, \dots, k_n)$.

It follows, that the intersection $D(k_1,\dots,k_{n+1})\cap|C(k_{n+1})|$ looks as claimed. Furthermore, for any zero-simplex $|s|$ of $\partial D(k_1,\dots,k_{n+1})$, the faces in the intersection of $|C(s)|$ with $D(k_1,\dots,k_{n+1})$ are again consecutively adjacent, and $D(k_1,\dots,k_{n+1})\setminus|C(s)|$ is edge-path connected. This completes the inductive argument and proves the lemma.

□

Proof of Main Theorem 37. The first part of the statement follows by construction.

For the second part, let K be a combinatoric such that $|K|$ is simply connected and let $(\sigma,\delta)\in\mathcal{T}_K$. By the definition of $\mathcal{T}_K$ the vector σ has as entries the angles of a flat-cone surface $\ell\colon K_1\to\mathbb{R}_+$ triangulated by K. By Lemma 11, ℓ is determined uniquely from σ and the choice of one of its edgelength. Choose a one-simplex $\{ij\}\in K_1$ and set the length of $\{ij\}$ equal to $\ell_{ij}=1$. Then, Lemma 11 determines ℓ from σ and $\ell_{ij}=1$ by $\varphi_{ij}(\sigma,1)=(\sigma,\ell)$. In the following we construct a polyhedron P with surface angles σ, dihedral angles $\delta_P\equiv\delta \mod 2\pi$ and edgelengths ℓ.

Let $k_1=i$. We construct a simplexwise affine and simplexwise injective map $P_1\colon|C(k_1)|\to\mathbb{R}^3$ as follows. By the definition of $\mathcal{T}_K$ and by Proposition 21 there is a polyhedral cone C_1 with underlying abstract simplicial cone $C(k_1)$ and surface angles σ^{k_1} and dihedral angles $\delta_{C_1}\equiv\delta^k \mod 2\pi$. Send $|k_1|$ under P_1 to the tip of C_1. Furthermore, send $|k|$ for every $k\in\operatorname{st}k_1$ under P_1 to the unique point on the edge $C_1(|k_1k|)$ of C_1 of distance ℓ_{k_1k} from $P_1(|k_1|)$. We have constructed the image of P_1 of all the zero-simplices of $|C(k_1)|$. Define P_1 on any two-simplex $|sk_1t|$ of $|C(k_1)|$ as the affine map defined by the images under P_1 of its zero-simplices $|s|$, $|k_1|$ and $|t|$. This also defines the image of P_1 of the one-simplices of $|C(k_1)|$. By the Side-Angle-Side Congruence Theorem for euclidean triangles the surface angles of P_1 are indeed the components of σ^{k_1} and its edgelengths are given by the restriction of ℓ to the one-simplices of $C(k_1)$. By constuction the dihedral angles of P_1 are $\delta_{P_1}\equiv\delta^k \mod 2\pi$. By construction P_1 is simpexwise affine and since the entries of ℓ are positive it is simplexwise injective.

By Lemma 38 there is a subset $\{k_1,\dots,k_N\}\subset K_0$ such that for each $n\in\{1,\dots,N\}$ the subsets $|D_n|:=D(k_1,\dots,k_n)$ of $|K|$ are simply connected, such that $|k_{n+1}|\in\partial|D_n|$ and such that $|D_N|=|K|$. Let now $n\in\{1,\dots,N-1\}$ and suppose there is a simplexwise affine and simplexwise injective map $P_n\colon|D_n|\to\mathbb{R}^3$ whose surface angles are given by the restriction of σ to the corners of $D_n:=\cup_{i=1}^n C(k_i)$, whose dihedral angles coincide up to 2π with δ on the one-simplices of D_n and whose edgelengths are given by ℓ restricted to the one-simplices of D_n.

In the same way as we contructed P_1 we construct a simplexwise affine and simplexwise injective map $P'_n\colon |C(k_{n+1})| \to \mathbb{R}^3$ whose surface angles are given by σ restricted to the corners of $|C(k_{n+1})|$, whose dihedral angles are δ modulo 2π restricted to the one-simplices of $|C(k_{n+1})|$ and whose edgelengths are given by ℓ restricted to the one-simplices of $|C(k_{n+1})|$. By Lemma 40 the intersection of $|C(k_{n+1})|$ with $D(k_1,\ldots,k_n)$ is given by

$$|C(k_{n+1})| \cap D(k_1,\ldots,k_n) = \cup_{l=1}^{m} f_l \cup \gamma_1 \cup \gamma_2,$$

where $f_1,\ldots,f_m$ are consecutively adjacent faces and γ_1 and γ_2 are egdepaths starting at a vertex of f_1, and of f_m, respectively. The image of $f_1 \cup \ldots \cup f_m$ under P_n is a set of consecutively adjacent triangles in $\mathbb{R}^3$ with the same edgelengths and dihedral angles beween them, as the image of $f_1 \cup \ldots \cup f_m$ under P'_n. It follows, that $P_n(f_1 \cup \ldots \cup f_m)$ and $P'_n(f_1 \cup \ldots \cup f_m)$ are congruent in $\mathbb{R}^3$. Let T be the euclidean isometry that takes the latter to the former and from now on also denote by P'_n the map $T \circ P'_n$.

Then, by construction P_n and P'_n coincide on $f_1 \cup \ldots \cup f_m$. We now argue, that P_n and P'_n also coincide on γ_1 and γ_2. If γ_1 and γ_2 are the empty set, or if they are contained in $f_1 \cup \ldots \cup f_m$, then the images of P_n and P'_n coincide on γ_1 and γ_2. Otherwise, consider the vertex $|k|$ of f_1 in which γ_1 starts suppose that $|l_1|$ is the adjacent vertex to $|k|$ on γ_1 that is not contained in $f_1 \cup \ldots \cup f_m$. Then the union of the image $P_n(|C(k)| \cap D(k_1,\ldots,k_n))$ and the image $P'_n(|C(k)| \cap |C(k_{n+1})|)$ is a set of consecutively adjacent triangles in $\mathbb{R}^3$ starting in the line segment $P_n(|kl_1|)$ and ending in the line segment $P'_n(|kl_1|)$. It is uniquely determined by the edge $P_n(|k\ell_1|)$, the angles of the triangles and the dihedral angles between any two consecutive triangles. However, the faces of

$$C := P_n(|C(k)| \cap D(k_1,\ldots,k_n)) \cup P'_n(|C(k)| \cap |C(k_{n+1})|)$$

have by construction the surface angles and dihedral angles beween adjacent faces of a polyhedral cone. Therefore, C defines a polyhedral cone and its first and last edges $P_n(|kl_1|)$ and $P'_n(|kl_1|)$ coincide. We repeat the argument for the next edge of γ_1 and sucessively for the remaining edges of γ_1 and obtain that P_n and P'_n coincide on γ_1. The same argument works for γ_2.

Define $P_{n+1}\colon D(k_1,\ldots,k_{n+1}) \to \mathbb{R}^3$ as P_n on $D(k_1,\ldots,k_n)$ and as P'_n on $|C(k_{n+1})|$. By the previous arguments the map P_{n+1} is well defined. By construction it is simplexwise affine, simplexwise injective, and has surface and dihedral angles given by the maps σ restricted to the corners of $D(k_1,\ldots,k_{n+1})$, and δ restricted to $D(k_1,\ldots,k_{n+1})$ and componentwise taken modulo 2π. □

Remark 41. Note that in the above construction the last two faces are implied as soon as the surrounding ones have been constructed. In particular, the equations $\widetilde{g}_K^{\mathrm{in}}(\sigma,x)(\{ijk\}) = 0$ for the two corresponding two-simplices $\{ijk\}$ are redundant, and may be dropped from the definition of $\widetilde{\mathcal{T}}_{K,\tau}$.

4.2 Dimension of the manifold of geometric realizations

In the following we indicate how to recompute the dimension of $\mathcal{T}_K$ modulo the Elimination Pattern Conjecture 34.

Theorem 42 (modulo EPC 34). *Let K be a combinatoric such that $|K|$ is simply connected. Then $\widetilde{\mathcal{T}}_{K,\tau}$ is an analytic submanifold of $\Omega_{K,\tau} \times \mathbb{R}_+^{K_1}$ of dimension*

$$\dim \widetilde{\mathcal{T}}_{K,\tau} = E.$$

Proof modulo EPC 34. By definition $\widetilde{\mathcal{T}}_{K,\tau}$ is the locus of $\Omega_{K,\tau} \times \mathbb{R}_+^{K_1}$ where

$$\widetilde{g}_K^{\mathrm{in}}(\sigma,x) = 0 \quad \text{and} \quad \widetilde{g}_{K,\tau}^{\mathrm{le}}(\sigma,\delta,\alpha,\beta,\gamma,c) = 0. \tag{23}$$

By Remark 41, there are two adjacent two-simplices $\{ijk\}$ and $\{jkl\} \in K_2$, such that if we remove the equations

$$\widetilde{g}_K^{\mathrm{in}}(\sigma,x)(\{ijk\}) = 0 \quad \text{and} \quad \widetilde{g}_K^{\mathrm{in}}(\sigma,x)(\{jkl\}) = 0$$

from the system of equations 23, then the new system of equations still defines the same set of solutions $\widetilde{\mathcal{T}}_{K,\tau}$. Define a new function

$$f_K^{\mathrm{in}} \colon (0,\pi)^{C_K} \times \mathbb{R}_+^{K_1} \longrightarrow (\mathbb{R}^3)^{K_2 \setminus \{\{ijk\},\{jkl\}\}}$$

by

$$f_K^{\mathrm{in}}(\sigma,x)(\{k\}) := \widetilde{g}_K^{\mathrm{in}}(\sigma,x)(\{k\}),$$

and a function

$$f_K \colon \Omega_{K,\tau} \times \mathbb{R}_+^{K_1} \longrightarrow (\mathbb{R}^3)^{K_2} \times_{k\in K_0} \mathbb{R}^{3(n_k-2)+n_k}$$

by

$$f_K(\sigma,\delta,\alpha,\beta,\gamma,c,x)(\{k\}) := \begin{pmatrix} f_K^{\mathrm{in}}(\sigma,x) \\ \widetilde{g}_{K,\tau}^{\mathrm{le}}(\sigma,\delta,\alpha.\beta,\gamma,c)(\{k\}) \end{pmatrix}.$$

Then, by the above $\widetilde{\mathcal{T}}_{K,\tau} = f_K^{-1}(0)$. By the Elimination Pattern Conjecture 34 there is an admissible coloring (C,A) of K^* that satisfies the conclusions of the Lemma 34 and such that its colored corners C are the corners of $\{ijk\}^*$, together with the corners of $\{jkl\}^*$. By Proposition 25 and by its proof $\mathrm{D}\,\widetilde{g}_{K,\tau}^{\mathrm{le}}$ is of full rank and furthermore it has a maximal submatrix M of full rank so that the columns of M contained in $\frac{\partial f_K}{\partial \sigma}$ are the partial derivatives $\frac{\partial f_K^{\mathrm{in}}}{\partial \sigma_{rst}}$ where

(rst) are the corners of either $\{ijk\}$ or of $\{jkl\}$. It folows that M and $\mathrm{D}\, f_K^{\text{in}}$ as submatrices of $\mathrm{D}\, f_K$ share no columns of $\mathrm{D}\, f_K$. Since by Proposition 9 we have that $\mathrm{D}\, f_K^{\text{in}}$ is of full rank on $(f_K^{\text{in}})^{-1}(0)$, we conclude that $\mathrm{D}\, f_K$ is of full rank on $f_K^{-1}(0)$.

By the Preimage Theorem [7] it follows that $\widetilde{\mathcal{T}}_{K,\tau}$ is a smooth submanifold of $\Omega_{K,\tau} \times \mathbb{R}_+^{K_1}$ of dimension

$$\dim \widetilde{\mathcal{T}}_{K,\tau} = \dim \left(\Omega_{K,\tau} \times \mathbb{R}_+^{K_1}\right) - \operatorname{rk} \mathrm{D}\, f_K.$$

We compute

$$\dim \left(\Omega_K \times \mathbb{R}_+^{K_1}\right) = \sum_{k \in K_0} (3(n_k - 2) + (n_k - 3)) + 3F + 2E,$$

and

$$\begin{aligned} \operatorname{rk} \mathrm{D}\, f_K &= \operatorname{rk} \mathrm{D}\, f_K^{\text{in}} + \operatorname{rk} \mathrm{D}\, \tilde{g}_{K,\tau}^{\text{le}} \\ &= (\operatorname{rk} \mathrm{D}\, \tilde{g}_K^{\text{in}} - 6) + \operatorname{rk} \mathrm{D}\, \tilde{g}_{K,\tau}^{\text{le}} \\ &= 3F - 6 + \left(\sum_{k \in K_0} (3(n_k - 2) + (n_k - 3)) + 3V \right) \end{aligned}$$

and obtain

$$\begin{aligned} \dim \widetilde{\mathcal{T}}_{K,\tau} &= 3F + 2E - (3F + 3V - 6) && (24) \\ &= (E - 3V) + E + 6 && (25) \\ &= -6 + E + 6 && (26) \\ &= E, && (27) \end{aligned}$$

where Equation 26 holds by Euler's Polyhedron Formula applied to V, and because in every simplicial two-complex K we have that $2E = 3F$. □

Theorem 43. *[modulo EPC 34] Let K be a combinatoric. For every $\{ij\} \in K_1$ there is a diffeomorphism*

$$\psi_{ij} \colon \mathcal{T}_K \times \mathbb{R}_+ \longrightarrow \widetilde{\mathcal{T}}_{K,\tau}.$$

In particular, if $|K|$ is simply connected, then $\mathcal{T}_K$ is a real analytic submanifold of $(0,\pi)^{C_K} \times \mathbb{R}_+^{K_1}$ of dimension

$$\dim \mathcal{T}_{K,\tau} = E - 1.$$

Proof modulo EPC 34. The projection map

$$\begin{aligned} \pi_{ij}\colon \widetilde{\mathcal{T}}_{K,\tau} &\longrightarrow \mathcal{T}_K \times \mathbb{R}_+ \\ (\sigma, \delta, \alpha, \beta, \gamma, c, x) &\longmapsto (\sigma, \delta, x_{ij}) \end{aligned}$$

is smooth. Let $(\sigma, \delta, \ell) \in \mathcal{T}_K \times \mathbb{R}_+$. Then, for every $k \in K_0$ there are $(\alpha_k, \beta_k, \gamma_k, c_k)$ in an appropriately projected domain, such that

$$(\sigma^k, \delta^k, \alpha_k, \beta_k, \gamma_k, c_k) \in \widetilde{\mathcal{T}}_{n_k}.$$

Recall from the proof of Proposition 23 that α_k, β_k, γ_k and c_k can be expressed as smooth functions of σ^k and δ^k. Hence α, β, γ and c can be expressed as smooth functions $\alpha(\sigma, \delta)$, $\beta(\sigma, \delta)$, $\gamma(\sigma, \delta)$ and $c(\sigma, \delta)$ of σ and δ. Then the function

$$\begin{aligned} \psi_{ij}\colon \mathcal{T}_K \times \mathbb{R}_+ &\longrightarrow \widetilde{\mathcal{T}}_{K,\tau} \\ (\sigma, \delta, \ell) &\longmapsto (\sigma, \delta, \alpha(\sigma, \delta), \beta(\sigma, \delta), \gamma(\sigma, \delta), c(\sigma, \delta), d_{ij}(\sigma, \ell)), \end{aligned}$$

where d_{ij} is the function defined in the proof of Proposition 11, defines a smooth inverse for π_{ij}. □

References

[1] A. D. Alexandrov. *Convex polyhedra.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. Translated from the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze and A. B. Sossinsky, With comments and bibliography by V. A. Zalgaller and appendices by L. A. Shor and Yu. A. Volkov.

[2] Marcel Berger. *Geometry II.* Universitext. Springer-Verlag, Berlin, 2009. Translated from the 1977 French original by M. Cole and S. Levy, Fourth printing of the 1987 English translation [MR0882541].

[3] Raoul Bricard. Mémoire sur la théorie de l'octaèdre articulé. *Journal de mathématiques pures et appliquées*, 1897.

[4] Augustin-Louis Cauchy. Sur les polygones et les polyèdres : second mémoire. *Journal de l'École polytechnique*, 1813.

[5] Robert Connelly. An immersed polyhedral surface which flexes. *Indiana Univ. Math. J.*, 25(10):965–972, 1976.

[6] Robert Connelly. A counterexample to the rigidity conjecture for polyhedra. *Inst. Hautesc Études Sci. Publ. Math.*, (47):333–338, 1977.

[7] Victor Guillemin and Alan Pollack. *Differential topology.* AMS Chelsea Publishing, Providence, RI, 2010. Reprint of the 1974 original.

[8] Jaques Hadamard. Erreurs des mathématiciens. *L'Intermédiaire des Mathématiciens*, 14, 1907.

[9] Michael Kapovich and John J. Millson. Hodge theory and the art of paper folding. *Publ. Res. Inst. Math. Sci.*, 33(1):1–31, 1997.

[10] Hermann Karcher. Remarks on polyhedra with given dihedral angles. *Comm. Pure Appl. Math.*, 21:169–174, 1968.

[11] Henri Lebesgue. Démonstration complète du theorème de Cauchy sur l'égalité des polyèdres convenxes. *L'Intermédiaire des Mathématiciens*, 16:113–120, 1909.

[12] Rafe Mazzeo and Grégoire Montcouquiol. Infinitesimal rigidity of cone-manifolds and the Stoker problem for hyperbolic and euclidean polyhedra. *J. Differential Geom.*, 87(3):525–576, 2011.

[13] A. D. Milka. On a certain conjecture of Stoker. *Ukrain. Geometr. Sb. Vyp.*, 9:85–86, 1970.

[14] A. V. Pogorelov. On a problem of Stoker. *Dokl. Akad. Nauk*, 385(1):25–27, 2002.

[15] Jürgen Richter-Gebert. *Realization spaces of polytopes*, volume 1643 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1996.

[16] William P. Thurston. Shapes of polyhedra and triangulations of the sphere. In *The Epstein birthday schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 511–549. Geom. Topol. Publ., Coventry, 1998.

[17] Walter Whiteley. How to describe or design a polyhedron. *Journal of Intelligent and Robotic Systems*, 11:135–160, 1994.